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INVESTMENT UNDER UNCERTAINTY: THE ROLE OF INVENTORY DYNAMICS

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ABSTRACT

Finished-good inventory is very common under market uncertainty. We built a continuous-time model to study how the inventory will impact a firm's value and investment decisions. Our model shows that the value of a company that followed the optimal inventory policy can be significantly higher than the traditional non-inventory company, particularly if the inventory-holding cost is not large. This premium becomes smaller as the holding cost is increased and is larger when demand is volatile, and when price elasticity is large. We also show that the optimal investment size can be significantly larger than the traditional non-inventory firm, particularly when the inventory-holding cost is low, demand volatility is high, and price elasticity is low. This paper develops a simulation algorithm to solve an iterative optimization problem in a path-dependent economy.

Keywords *inventory dynamics, real options, corporate investment.*

1. Introduction

A number of papers have used the contingent-claim model for valuation of companies and to identify optimal investment decisions under uncertainty. In these papers, uncertainty is introduced by means of a stochastic underlying variable such as revenue or output price or demand strength, which follows an exogenously specified random process. The firm's valuation and the investment decisions are based on this uncertain state variable.

These papers all assume that the firm sells all of its output when it is produced and that there is no possibility of maintaining any inventory of the output; thus, they ignore any effect of inventory management on firm performance. Examples include Huberts et al. (2015), Jou and Lee (2008), Mauer and Ott (2000), Miao (2005), Dangl (1999), Bar-Ilan and Strange (1999), and many others. However, empirical evidence indicates that inventory management does in fact have a significant impact on the firm's performance, for example, Basu and Wang (2011), Elsayed (2015), Koumanakos (2008), Kroes and Manikas (2018), and Ndubuisi et al. (2020). In this paper, therefore, we address the question: how does (optimal) inventory management impact the value and the optimal investment decision of the firm?

A couple of papers have looked at the relationship between inventory and capital investment. Pindyck (1982) examines the impact of uncertainty on investment (capital stock) with and without the possibility of output inventory. However, his paper looks at incremental adjustments to capital stock and is therefore an adjustment-cost model; he shows that the directional impact of uncertainty on capital is the same with and without inventory, although the size (sensitivity) of the adjustment is smaller with inventory. Kim (2020) shows that a firm's inventory dependence (i.e., the importance of inventory in the firm's operations) makes capital investment less sensitive to important economic measures such as firm performance, industry growth, and uncertainty.

The other papers with inventory are limited to raw-material or work-in-process inventory, not output inventory, as in our paper. For instance, in Bianco and Gamba (2019), the firm uses input inventory as an operational hedge for risk management to mitigate the price risk of input materials. Cortazar and Schwartz (1993) examine the valuation of a company with a two-step manufacturing process with a work-in-process inventory.

Our paper examines a firm that has the ability to maintain inventory of the output by comparing it with the traditional model's no-inventory firm. We focus on how much this ability to maintain inventory is worth when used optimally, how this premium in value (relative to the no-inventory firm) is affected by various economic parameters, and how the ability to maintain inventory impacts the firm's investment (size and timing) decision. The firm's policy might be set so as to maximize the current profit or firm's value (because both seem to be used in practice), and we consider both scenarios.

The main results are as follows. First, if inventory holding cost is small, then the firm value with inventory can be substantially larger than in the traditional (no-inventory) models. Second, the behavior of the firm's value with respect to investment size is quite different for a firm with inventory and one without inventory; hence, the conclusions of the traditional literature with regard to investment size might have to be modified for the realistic situation of a firm with inventory. Third, if an inventory-carrying firm's inventory policy is based on maximizing profits rather than value, it might end up with a firm's value that is below the benchmark (no-inventory) firm, particularly if the demand elasticity is large and the demand level is small; thus, maximizing the firm's profits might end up causing destruction of the firm's value.

The rest of the paper is organized as follows. Section 2 describes and derives the model with and without the ability to maintain inventory. Section 3 presents the results of the model. Section 4 summarizes and concludes.

2. The Model

A firm has a production facility whose size is given by the amount of capital K. This facility allows it to produce Q units of the output per unit time, where Q is given by $K = Q^{1/\delta}$; the exponent δ can be viewed as the returns-to-scale of the technology used. The output is sold in the product market, at a price given by

$$p = \theta - \gamma q, \tag{1}$$

where q is the amount sold, θ is the random/stochastic strength of demand (or demand shock), and γ is the sensitivity of the price to the amount sold (or the elasticity of demand). This price process is commonly used in the literature to represent the output's demand curve (Aguerrevere 2003; Dangl 1999; Huberts et al. 2015). The strength of demand θ introduces uncertainty in the model and is assumed to follow the lognormal process:

$$d\theta = \mu\theta dt + \sigma\theta dZ, \tag{2}$$

where μ and σ are the trend and volatility of the demand process and Z is a standard Wiener Process.

We assume that the plant always operates at full capacity, that is, the production rate is always Q, as in, for example, Bar-Ilan and Strange (1999) and Dangl (1999). This is partly because of analytical tractability; if the firm could vary the output rate, it would make the analysis much more complicated. However, this is also a common modeling assumption, because it is a reasonable description of many real-world process industries such as paper, chemicals, etc. (Lederer and Mehta 2005). Moreover, it is consistent with the "price postponement with clearance" argument of Van Mieghem and Dada (1999). Finally, in many industries the firms make the production plans before the actual realization of market demand, and many firms find it difficult to produce below full capacity because of commitments to suppliers and because of fixed costs associated with flexibility (Goyal and Netessine 2007).

Finally, the cost of investing in the production plant has both a constant component and a component that is increasing linearly in the investment size (amount of capital, K). Let this investment cost be $I(Q) = m_0 + m_1 K = m_0 + m_1 Q^{1/\delta}$, where m_0 is the fixed investment cost and m_1 is the variable cost of investment per unit of capital (recall that capital is $K = Q^{1/\delta}$).

Because we are studying the effect of maintaining output inventory, we study both cases: the firm with and without inventory. We call the latter the "inflexible" or "benchmark" firm because it is this type of firm that has been examined in the literature so far. Also, a firm with the ability to maintain inventory might have one of two objectives when making its inventory decisions; its objective could be to maximize either the firm's profit for the period or the value of the firm. A quick look at the production economics literature indicates that, in many (if not most) cases, the objective is specified as profit-maximization; however, in the finance literature, the objective is value-maximization. We, therefore, examine both cases for the inventory-holding firm.

For high price-sensitivity γ, the profit-maximizing firm might sell a smaller amount to keep prices high (so as to maximize the current profit); however, this will increase the future inventory level and drive up inventory costs and thereby reduce the firm's value. Because it acts myopically in ignoring future costs when making decisions, we call the profit-maximizing firm a "myopic" firm. The value-maximizing firm, however, behaves strategically in considering the overall effect of its decisions on the value of the firm (taking into account all costs), hence we call it a "strategic" firm. Thus, we analyze three different firms: (i) benchmark firm (no inventory), (ii) myopic firm (with inventory), and (iii) strategic firm (with inventory).

2.1. Benchmark Firm Valuation

The benchmark firm produces and sells at a rate of Q units per unit time, that is, $q = Q$, then, its profit flow is given by

$$\pi(\theta) = pq - cq = (\theta - \gamma q)q - cq = \theta Q - (\gamma Q + c)Q \tag{3}$$

Then, the value of the plant is given by

$$V(\theta,\, Q) = E_0^Q\left[\int_0^T e^{-r\tau}\pi_\tau d\tau\right] = \frac{1 - e^{-(r-\mu)T}}{r - \mu}\theta Q - \frac{1 - e^{-rT}}{r}(\gamma Q + c)Q, \tag{4}$$

where T is the remaining life of the project.

This is the standard approach in the literature, and the above valuation is consistent with the existing models, for example, Bar-Ilan and Strange (1999). If the firm also chooses the size of the investment optimally, then it will maximize the above firm value at the time of investment less the investment cost, that is, $Q^* = \text{argmax}_Q\left\{V(\theta_0,\, Q) - \left(m_0 + m_1 Q^{1/\delta}\right)\right\}$, where θ_0 is the demand strength at the time of investment. This maximization gives the following:

$$Q^* = \frac{\theta_0 r(1 - e^{-(r-\mu)T})}{2\gamma(r - \mu)(1 - e^{-rT})} - \frac{c}{2\gamma} - \frac{r m_1 Q^{*\frac{1-\delta}{\delta}}}{2\gamma\delta(1 - e^{-rT})} \tag{5}$$

2.2. Myopic Firm Valuation

Suppose the existing inventory level is N and the inventory holding cost is k per unit of product per unit time, then the profit stream is given by the following:

$$\pi(\theta, N) = pq - cQ - kN = (\theta - \gamma q)q - cQ - kN \tag{6}$$

The goal of the myopic firm is to maximize the instantaneous profit, it will set $\frac{d\pi}{dq} = 0$, which gives

$$\theta - 2\gamma q^* - k\frac{dN}{dq} = 0, \quad \text{or} \quad q^* = \frac{\theta - k\frac{dN}{dq}}{2\gamma} \tag{7}$$

Note that the optimal q is a function of θ and N, for example, $q^* \equiv q^*(\theta, N)$. To simplify it, we note that, when one more unit is sold (i.e., q is up by 1), the inventory balance will decline by 1 (i.e., N will fall by 1), hence $\frac{dN}{dq} = -1$. This gives us the optimal amount that the myopic firm will sell at any instant:

$$q^* = \frac{\theta + k}{2\gamma} \tag{8}$$

Here, the instant sales q^* could be larger than capacity Q due to the inventory. However, there is an upper limit on how many units the firm can sell at any point in time; suppose it can draw down inventory at a rate of $\dot{N}\left(= \frac{dN}{dt}\right)$

units per unit time, then the sales amount is limited by $q \leq Q + \dot{N}$. The demand threshold where the sales amount reaches the upper limit is given by $\overline{\theta} = 2\gamma\left(Q + \dot{N}\right) - k$.

Moreover, please notice that from the inventory dynamic, we can solve its value up to time t

$$N_t = tQ - \int_0^t q_\tau d\tau \tag{9}$$

When substituting this into Equation (6), we can rewrite instantaneous profit as follows:

$$\pi(\theta) = (\theta–\gamma q)q - cQ - ktQ - k\int_0^t q_\tau^* d\tau \tag{10}$$

The valuation of the myopic firm cannot be expressed analytically, but the general approach is as follows: firm's value $V(\theta,\ t)$ is given by

$$V(\theta,\ t) = E_t^Q\left[\int_t^\infty e^{-r\tau}\pi\left(\tau,\ \theta|q_\tau^*\right)d\tau\right] \tag{11}$$

Under Feynman–Kac, a partial differential equation (PDE) can be written as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V}{\partial\theta^2} + \mu\theta\frac{\partial V}{\partial\theta} + \pi\left(t, \theta|q_t^*\right) = rV \tag{12}$$

which is subject to three boundary conditions:

$$V(\theta \downarrow 0, t) = 0 \tag{13}$$

$$V(\theta \uparrow \infty, t) = Q\left(\frac{\theta}{r - \mu} - \frac{\gamma Q + c}{r}\right) \tag{14}$$

$$V(\theta, T) = 0 \tag{15}$$

Equation (13) states that the firm's value approaches zero when demand falls to very low levels. Equation (14) states that, when demand is very large, the firm will sell its entire output and keep no inventory. Equation (15) states that, when the firm is liquidated (at time $t = T$), its liquidation value will be zero (remaining inventory value will be offset by storage costs).

Unfortunately, the PDE still cannot be solved directly because it is path-dependent, that is, current profit depends on previous inventory history. We will therefore turn to the Monte Carlo simulation to solve this; the numerical details are in Appendix A.

2.3. Strategic Firm Valuation

The strategic firm will select optimal sales *ex ante* to maximize the present value of all future profit flows under demand uncertainty. Mathematically, the general form of profit process for strategic firm is the same as that for the myopic firm (e.g., Equation (10)), although they differ in the choice of optimal sales q_t^*. However, the optimal sales level for the strategic firm is difficult to express explicitly because we do not yet know the expression for the firm's value *ex ante*, which in turn varies with q_t^*.

To begin with, the value of the strategic firm after investment, $V(\theta, t,\ q)$, is governed by following equation:

$$V(\theta, t,\ q) = \max_q E_t^Q\left[\int_t^\infty e^{-r\tau}\pi(\theta, \tau,\ q)d\tau\right] \tag{16}$$

It can be observed that the Equation (16) is similar to Equation (11) except for the search process of optimal q_t. It can be similarly reformulated to the following PDE:

$$\max_q\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V}{\partial\theta^2} + \mu\theta\frac{\partial V}{\partial\theta} + \pi(t, \theta)\right] = rV \tag{17}$$

Technically, the profit flow at any time t depends on previous optimal sales q_{t-1}^* (to calculate inventory), which also has to be solved in the entire PDE domain. Due to the complex nature of path-independence, we also turn to simulation to solve it. It has similar boundary conditions as for myopic firms:

When demand approaches zero, the firm's value becomes zero

$$V(\theta \downarrow 0, t) = 0 \tag{18}$$

When θ is very large, the firm will sell its full output, hence no new inventory is generated

$$V(\theta \uparrow \infty, t) = Q\left(\frac{\theta}{r-\mu} - \frac{\gamma Q + c}{r}\right) \tag{19}$$

Suppose there is a life limit $t = T$ such that the firm will be liquidated with zero

$$V(\theta, T) = 0 \tag{20}$$

These boundary conditions will help establish the accuracy and convergence of simulation results. The detailed algorithm is listed in Appendix B.

2.4. A brief discussion of the numerical method

The computation for the strategic firm is challenging because the firm's current (optimal) inventory is an outcome of past production and optimal sales, and the past optimal sales are linked to the current and future (optimal) inventories. There are two ways to solve for the optimal q_t^*, the "lumpy" approach and the "stepwise" approach. Under the lumpy approach, we forecast sales at all points in time $\boldsymbol{q_{1\sim T}} = (q_1, q_2, \ldots, q_T)$ by iterating over all possible sales combinations (e.g., sales could be any real non-negative numbers) to maximize the expected firm's value $E_0^Q(\theta_t, q_t^*)$. In the stepwise approach, we only consider solving optimal sales up to time t, $\boldsymbol{q_{t\sim T}} = (q_t, q_{t+1}, \ldots, q_T)$ to the maximize expected firm's value at current t: $E_t^Q(\theta_{t\sim T}, q_{t\sim T}^*)$ and iterate from $t = T$ to 1. From our numerical results, there does not seem to be much of a difference between the two methods in computation time.

However, simulation is costly in computation time, which makes it less suitable for the strategic-type firm. In addition, a longer time series (e.g., a longer firm life) will exponentially increase central processing unit (CPU) time. For example, if we simply search for optimal sales at each time spot to maximize the firm's value at entry, it may cost several minutes to a half hour to find the solution for a short firm life, even for a single path! Such a method cannot solve the problem for a large batch of simulation paths (say, 10,000 paths).

Our proposed searching algorithm can be described as follows:

Step 1. To start with, we assume that all periods of inventories are positive and compute the corresponding optimal sales, then we recalculate the updated inventories. If all of them are positive, then it is one of our solutions, if not, go to step 2.

Step 2. Suppose the first period of zero inventory is $t = i$, then we check if the inventories of all previous periods, for example, $t = 2, 3, \ldots, i-1$, are positive. If not, then we move the search to $t = i + 1$ and redo this part; if yes, then do the following:

1. Assume all inventories after $t = i$ are positive, and update if under such assumption both $N_i = 0$, and $N_{i+1,\ldots T} > 0$ (e.g., all assumptions are valid). If yes, we find the solution. If not, we do next.
2. Then we will have two possible outcomes:

   If $N_i > 0$, then our assumption on $N_i = 0$ is invalid and we move a new search to $t = i + 1$, the reason is that the assumption of all positive future inventories has maximized q_i (e.g., minimize N_i).

   If $N_i = 0$, but not all updated future inventories are positive, then it indicates that there should be a second period, $t = j\,(j > i)$, at which $N_j = 0$, for example, repeat step 2 to find out j period.

Step 3. Because step 2 could generate multiple solutions for $N_i = 0$ and/or $N_j = 0$, given $max(i,j) < T$, we need to repeat both steps to find all possible solutions until the end of the time series. To facilitate this process, we can, first, find the latest time when the inventory becomes zero, then all previous periods should also be candidates.

Although to prove the convergence and stability of our algorithm is beyond our capacity. It should not be hard to convince because we will have a finite set of solutions, for example, 2^N, there always exists the best set to maximize the equity value *ex ante*, in an iterative procedure that always improves value function, we should get the optimal solution in a certain number of iterations.

Under our proposed searching algorithm, the CPU time to calculate a single path can be decreased to 0.01 second, which is still very time-consuming, particularly for comparative static analysis or searching for optimal investment decisions, thus we have to limit to a max firm life of 10 years with 1-year time step.[1] To enhance the accuracy, we then simulate 20 batches to get the standard error down to the 1% level.

2.5. Analysis

Before presenting the numerical results, we discuss, briefly, what we can expect from the computations. As discussed above, all the firms will produce at the same rate Q but will sell different amounts. The inflexible (benchmark) firm does not have the ability to maintain inventory, hence it will sell all that is produced (Q). Both the myopic and the strategic firm can sell at a different rate because they have the ability to maintain inventory; therefore, they will sell more than or less than the amount they produce (Q).[2] However, their objective will be different: the myopic firm wants to maximize profits every period, whereas the strategic firm wants to maximize the firm's value. Because its decisions are designed to maximize the firm's value, it is clear that the strategic firm's value will be the highest of the three types; the next should be the myopic firm because it also has flexibility with regard to sales but uses its flexibility to myopically maximize instantaneous profits. The inflexible (benchmark) firm should have the lowest value because it is not able to maintain inventory and hence has no flexibility with regard to sales. Thus, the usual ordering of the firm's value is strategic, myopic, and benchmark, in order of decreasing value. There, however, is one exception, discussed below.

If the myopic firm sells a smaller quantity (i.e., q^* is smaller), then it will build up more inventory, which might destroy the firm's value because of inventory costs (recall that the myopic firm is maximizing profits, not the firm's value). Therefore, because of the excess inventory carrying cost, it is possible (although perhaps counterintuitive) that, when q^* is small enough, the value of the myopic firm falls below the value of the benchmark firm. Recall that the myopic firm sells the output at the rate of $q^* = \frac{\theta+k}{2\gamma}$; hence q^* is increasing in θ and k, while it is decreasing in the demand elasticity γ. Therefore, the myopic firm's value is more likely to be below the benchmark value when θ and k are small and when γ is large; alternatively, the myopic firm's premium is less likely to be negative when the demand level (θ) is large and the demand elasticity (γ) is small. Both of these are confirmed by our numerical results below.

Thus, the premium for the strategic firm's value over the benchmark firm will always be positive. However, the value of the ability to keep inventory will decline as the inventory holding cost rises, thus the premium should be decreasing in k (and approaching zero for a large enough k). However, the premium for the myopic firm over the benchmark firm can be positive or negative. Moreover, for all cases (positive or negative), this premium should approach zero for a large enough k because the ability to maintain inventory will make no difference if inventory holding costs are very high.

3. Numerical Results from the Model

3.1. Base-Case Parameter Values

In this section, we present numerical results from the simulation. We start with a comparison of the firm's value as a function of inventory holding cost k, for the myopic firm and the strategic firm as well as the benchmark firm (for the benchmark firm, the value will obviously be independent of k). For numerical results, we need to specify the input variables. We use a set of "base case" values for the input variables, as follows: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0, m_1 = 5$, $K = 8$, $T = 10$, $k = 0.2$, and $\theta_0 = 6$. We choose them based on the following economic literature:

For the interest rate, we adopted r = 4% as in Aretz and Pope (2018), in line with the average 10-year U.S. treasury rate. For the expected drift of the output price, we use $\mu = 0$, which conforms to Bayer (2007) and Lambrecht (2001). The cash flow volatility σ is set at 0.2, in following Lyandres and Zhdanov (2013) and Arnold (2014). The return-to-scale parameter δ is set to 0.5, the approximate average of past estimates.[3] All others are structural parameters and can be studied in a comparative statics analysis.

[1]In fact, MATLAB only allows a minimum time step = 0.5 year for such simulation because it only can iterate max 2^{20} inventory policies for a single simulated path. However, the smaller step size will not only cost much more time but might also make the simulation results non-convergent.

[2]Note that it can sell more than it produces only if there is inventory on hand.

[3]The return to scale varies in a large range in the past papers to entertain corresponding calibration performance. However, varying its value does not alter our main conclusion, so we take an approximately average of past values, to name a few, they are Miao (2005) ($\gamma = 0.4$), Danis and Gamba (2018) ($\gamma = 0.475$), and Riddick and Whited (2009) ($\gamma = 0.75$).

3.2. Valuation and Premium over the Benchmark Firm

3.2.1. Base case results

The firm's value as a function of inventory holding cost k for the three firms, shown in Figure 1(a): benchmark or inflexible firm (black line), myopic firm (blue line), and strategic firm (broken red line), and the same results in terms of premium over the benchmark firm's value are shown in Figure 1(b). As expected from the above discussion, the benchmark firm's value is independent of k and the strategic firm's value is the highest over the entire range. Also, the value of the strategic firm is a decreasing function of k; for a small k, the difference in the firm's value is substantial (about 11% above the benchmark firm's value for $k = 0$); but the difference falls rapidly with k, and it becomes negligible levels for k that exceeds 1.5. This is not surprising because the inventory becomes more expensive to support as k is increased, hence the ability to maintain inventory becomes less valuable; thus, the strategic firm value (and premium) is decreasing in k.

The behavior of the myopic firm value is a little different. First, it is a U-shaped function of k, falling from 6.75% for $k = 0$ to –12.1% for $k = 2$ and then rising slightly as k is increased further. Second, the myopic firm's value falls below the benchmark firm value when k is large enough (for k exceeding 0.4).

Thus, for a small inventory holding cost, the myopic firm value can also be significantly higher than the benchmark firm's value. However, for a larger inventory holding cost, the ability to maintain inventory can destroy value if it is not used optimally, for example, if maximizing short-term profit instead of value as in the myopic firm; in the base case, up to 12% of the firm's value can be destroyed this way. The U-shaped relationship can be explained as follows. As k is increased from 0, there are two opposing effects on the firm value: (i) direct effect: a higher k means higher inventory holding cost, which lowers the firm value and thus results in a downward-sloping curve; and (ii) indirect effect through q^*: as discussed above, a higher k results in higher q^*, that is, the firm starts reducing inventory; the resulting lower inventory holding cost will increase firm value and this will result in an upward-sloping curve. The latter effect dominates for a larger k, giving rise to the overall U-shaped relationship between inventory holding cost and the myopic firm's value, observed in Figure 1.

There are two points worth noting in these results. First, the benefits of maintaining inventory diminish rapidly as the inventory holding cost rises. Therefore, in industries where the inventory holding costs are high, it would make sense to use more inventory-minimizing techniques such as JIT (just in time). Second, an inventory management policy of choosing inventory levels based on maximizing profits (as in the myopic firm) could, in fact, end up reducing the firm's value relative to an inflexible (a no-inventory) firm. This is an important point because, in practice, inventory policy is often implemented out by operating managers whose objective is to maximize annual profits (Bassamboo et al. 2020; Canyakmaz et al. 2022; Li et al. 2021; Ma et al. 2022; Transchel et al. 2022; Zhao 2008). Our result indicates that profit-maximizing might not, in fact, be optimal because it could result in reduced firm value (particularly for large inventory holding cost). This suggests that it is better for inventory policy to be set on the basis of the firm's value rather than profits.

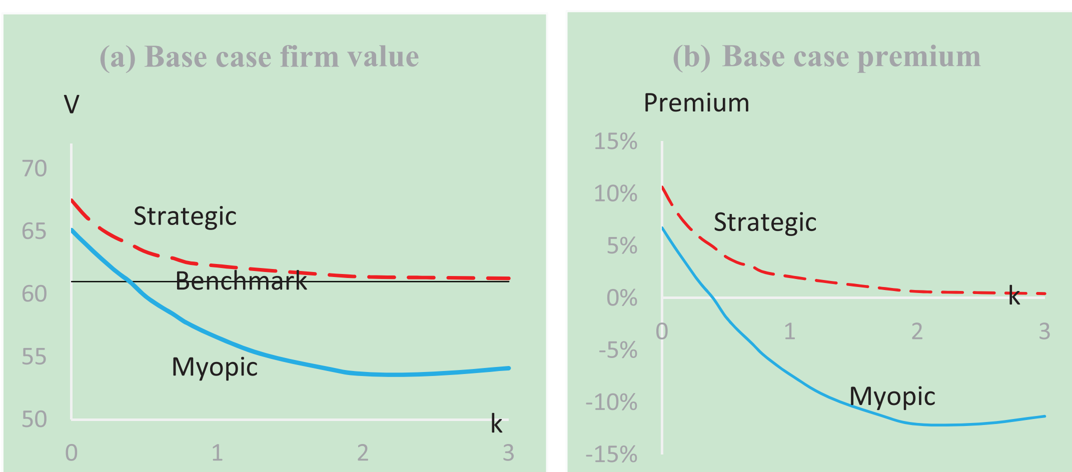

Figure 1: Shows the firm values for the three firm types (benchmark, myopic, and strategic) and the premium over the benchmark for the myopic and strategic firms. The base-case parameter values are used: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, $K = 8$, $\mathrm{T} = 10$, and $\theta_0 = 6$.

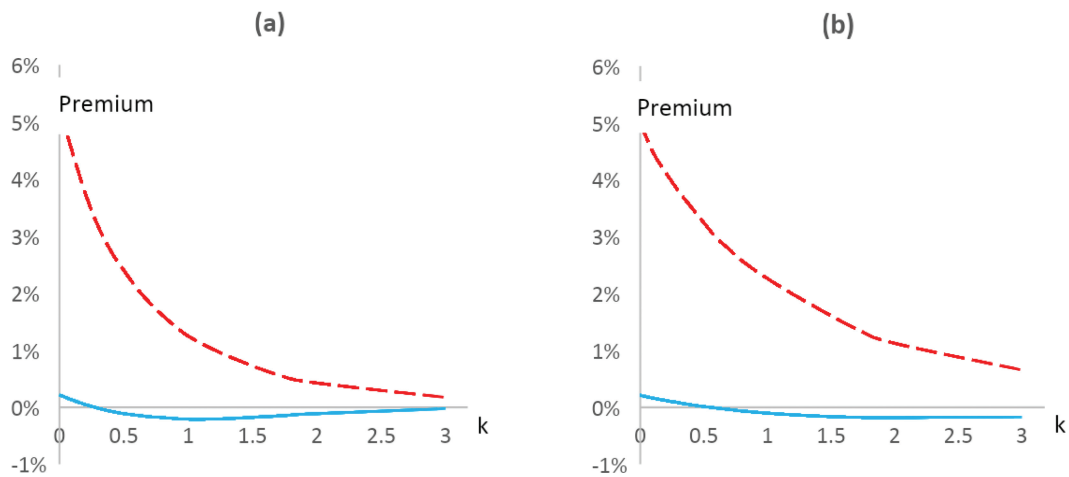

Figure 2: Shows the premium over the benchmark firm for two special cases, $\gamma = 0.5$ and $\theta_0 = 12$ (base-case value for all other parameters). The broken red line shows the strategic firm, and the solid blue line shows the myopic firm. Apart from the above parameters, the base-case values are used: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, $K = 8$, $T = 10$, and $\theta_0 = 6$.

Next, recall from our discussion above that the myopic firm's premium is less likely to be negative when demand level (θ) is large and demand elasticity (γ) is small because q^* will be larger and the firm will be less likely to accumulate large quantities of inventory, and, with lower inventory holding costs, there will be less value destruction. Thus, for high θ and/or low γ, we would expect the negative premium in the myopic firm's value to be smaller than in the base case. Repeating the numerical computations with these two scenarios ($\theta = 12$ and $\gamma = 0.5$), we find that this is indeed the case. As Figure 2 shows, in both cases, the negative premium that we noted in Figure 1(b) becomes much smaller; in fact, the premium is very close to zero. This is because, with a larger q^*, the myopic firm will be selling more and leaving less in inventory, which makes it resemble more and more the benchmark firm; thus, with a high θ and low γ, the difference between myopic and benchmark firms will shrink significantly and the premium will be closer to zero, consistent with Figure 2.

3.2.2. Comparative statics

Because there is no inventory with the benchmark firm, the benchmark firm has to sell everything it produces, even if it means selling at low prices during low-demand periods. However, both the myopic and the strategic firms can maintain inventory rather than selling the output at low prices, the former doing it in such a way as to maximize (short-term) profit, whereas the latter maximizes the firm's value. Thus, as explained in Section 2.4, both firms will generally be valued at a premium over the benchmark firm (although there might be exceptions for the myopic firm, as explained above). In this section, we find that this is indeed the case; the numerical results show that, for reasonable parameter values, the premium over the no-inventory firm is positive for both firms (Figure 3(a)–(g)).

We now look at how the various input parameters affect the premium for the myopic and strategic firms over the value of the benchmark no-inventory firm. Anything that reduces the need to maintain inventory (thereby reducing the value of the option to maintain inventory), for example, greater demand level, will move the premium closer to zero. Conversely, anything that makes the option more important (e.g., higher demand volatility) will move the premium away from zero.[4] This is indeed what our numerical results (below) confirm.

First, we look at the effect of demand volatility (σ). For both strategic and myopic firms, the premium is an increasing function of σ. As we know from the standard option theory, greater volatility increases option values. Because both firms have the option to maintain inventory, it is not surprising that the firm's value (and premium over the benchmark firm) is increasing in volatility in both cases, as shown in Figure 3(a).

Next, a higher demand growth rate (μ) has a negative effect on both strategic and myopic firm value, as shown in Figure 3(b), and both premiums approach zero when the growth rate is very large. This is also as expected because

[4]Recall that, in the base case (Figure 1), the premium for the myopic firm could be positive or negative, depending on the inventory holding cost k. For a positive (negative) premium, moving closer to zero implies that the premium will be decreasing (increasing); moving away from zero will imply just the opposite. Therefore, for the myopic firm, whether the premium is increasing or decreasing will depend on whether it is positive or negative. This is illustrated in Figure 4.

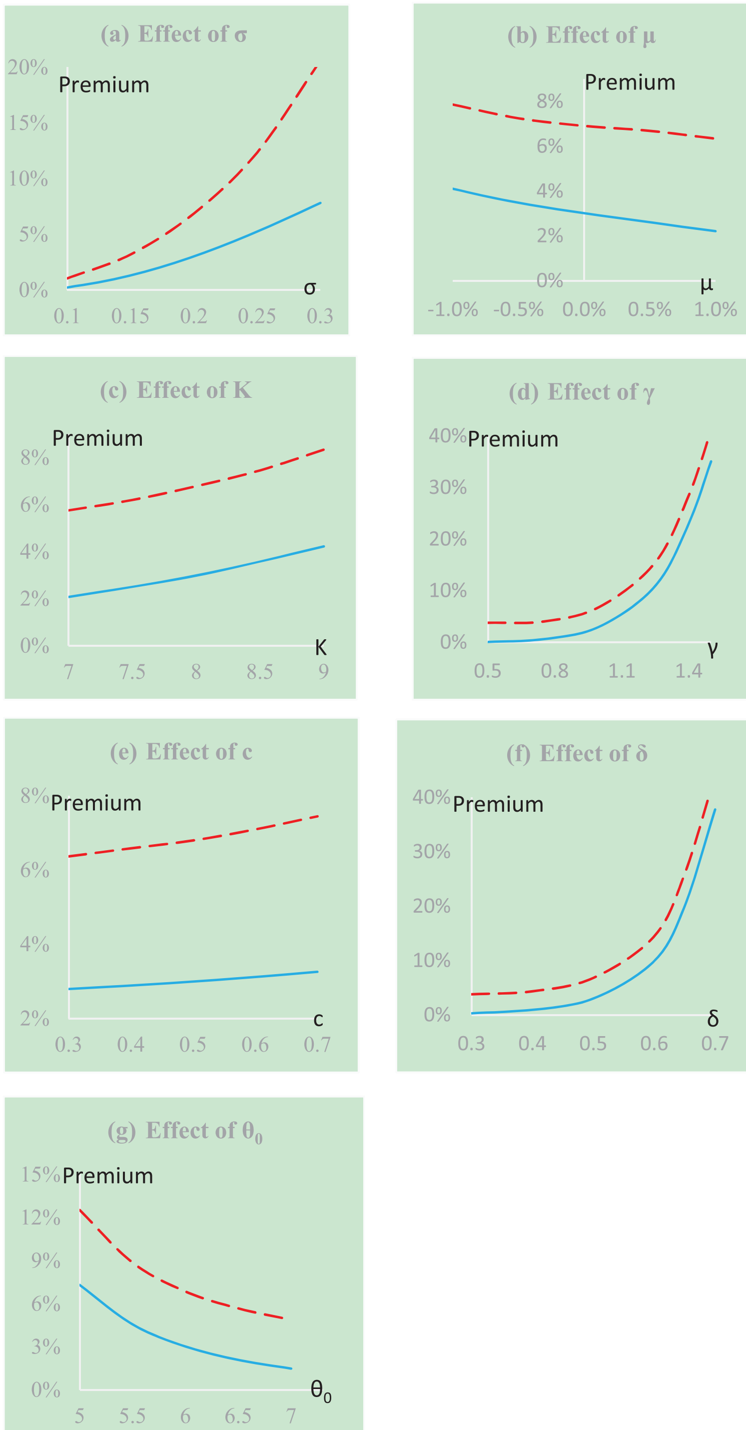

Figure 3: Comparative static results for the premium relative to the benchmark firm. The broken red line shows the strategic firm, and the solid blue line shows the myopic firm. The base-case parameter values are used: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, $K = 8$, $\mathrm{T} = 10$, $\mathrm{k} = 0.2$, and $\theta_0 = 6$.

a higher growth rate will cause both firms to sell more, hence inventory will play a diminished role and the premium will trend toward zero as a result (and they are decreasing functions of μ because both premiums are positive).

A larger capacity (K) means that the firm is producing more because it always produces at full capacity; when it is producing more, there will be more inventory, hence inventory will play a larger role. Thus, the magnitude of the premium (resulting from the ability to maintain inventory) will rise. As a result, premiums will move away from zero as K is increased, that is, because the premium is positive, it will increase with K in both cases. As we see in Figure 3(c), this is indeed the case.

Next, greater demand price sensitivity (γ) will result in a lower current output price (all else remaining unchanged), hence both flexible companies will sell less and therefore maintain higher inventory levels. This means that a higher γ will cause the inventory effect to be larger; hence the magnitude of the premium will be larger (i.e., it will move away from zero) as γ is increased. That is, the premium, if positive (negative), will be increasing (decreasing) in price sensitivity. As we see in Figure 3(d), this is indeed the case, with the premium for both firms being positive and increasing in γ.

For the operating cost (c), note that a higher c means that the margin is lower, which has the same effect as a lower price or a higher γ; thus, as in the case of γ, a higher c will result in the inventory becoming more important, hence the magnitude of the premium will increase with c, that is, the premium will move away from zero as c is increased or the premium, if positive (negative), will be increasing (decreasing) in c. As shown in Figure 3(e), this is exactly what we find, with the premium in both cases being positive and increasing in c.

A larger returns-to-scale parameter (δ) implies that a greater quantity will be produced with the same amount of capital, therefore, the ability to maintain inventory will be more valuable. Thus, the premium should be moving away from zero (if positive, an increasing function of δ), which is consistent with our numerical results shown in Figure 3(f).

When the demand level at investment (θ_0) is higher, both firms will sell more and thus have less in inventory; therefore, inventory will play a reduced role and the magnitude of the premium will fall, that is, the premium will move toward zero. As a result, a positive premium will be a decreasing function of θ_0. This is consistent with our numerical results, as shown in Figure 3(g). Finally, the two parameters interest rate (r) and investment cost (m_1) do not have a noticeable effect on the two premiums, because the myopic and strategic firms are impacted the same way as is the benchmark firm.

In the above comparative static results, the myopic firm premium was positive in all cases; with negative premium, the relationship will seem somewhat difference. To illustrate, we show in Figure 4 the effect of μ and γ when the myopic premium is negative (by setting inventory-holding cost $k = 0.5$ instead of 0.2). We note that (for the myopic firm premium) the effect is now different from the previous case (Figure 3), that is, increasing in μ and decreasing in γ; this is because the myopic premium is negative, as discussed above.

To summarize, the above comparative static results show that the premium over the benchmark firm's value can vary a lot when the input parameters are varied. Moreover, we also find (not shown) that the response of the firm's

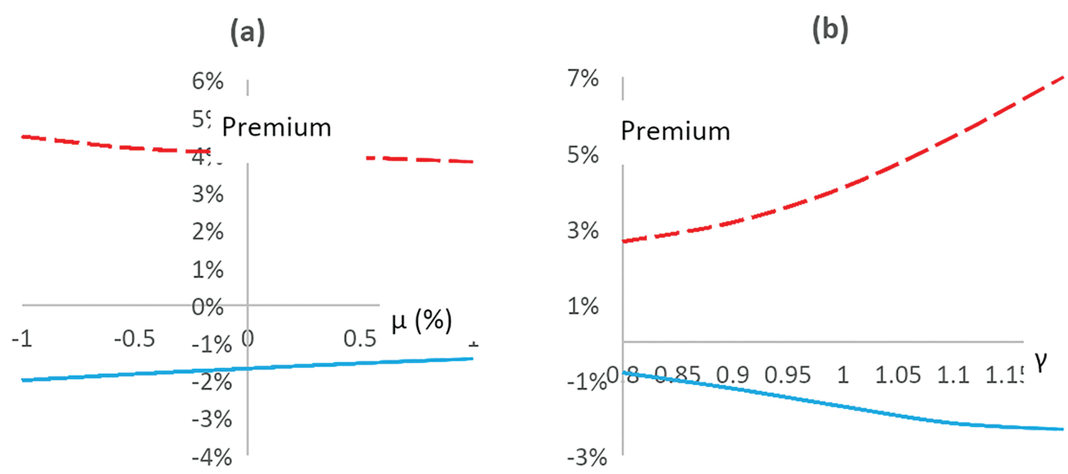

Figure 4: Some comparative static results for the premium *when the myopic firm's premium is negative*. The broken red line shows the strategic firm value, and the solid blue line shows the myopic firm value. The base-case parameter values are used: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, $K = 8$, $T = 10$, $k = 0.5$, and $\theta_0 = 6$.

value to certain parameter values (e.g., investment size) can be quite different for firms with inventory and those without inventory; the implication is that investment size and its sensitivity to various parameters can be quite different from the traditional models if we include the ability to maintain inventory.

3.3. Optimal Investment Size

Real-option models study the timing and size of investment (e.g., Bar-Ilan and Strange 1999; Besanko et al. 2010; Shibata and Nishihara 2018). However, as discussed by Li and Mauer (2016), firms often do not have the freedom to choose investment timing because external circumstances, such as competition or a short life of investment opportunity, determine the timing of investment. The value of such an investment opportunity can dissipate rapidly if not acted upon immediately, particularly in high-technology industries or even traditional industries with substantial competition. In such situations, the firm does not have the luxury of waiting for the optimal time to invest. However, the firm can in general choose the size or scale of the investment (Dangl 1999; Decamps et al. 2006; Dixit 1993). Therefore, the choice of investment size, given exogenously specified timing, is an important corporate decision; see, for instance, Jou and Lee (2004) or Moyen (2007).

In this section, we examine the investment size (or production capacity) decision and how it is impacted by the firm's ability to maintain inventory. The benchmark firm must sell everything it produces right away, hence it will be more hesitant to invest in a large capacity relative to the myopic firm or the strategic firm. Thus, the optimal investment size should be higher when the firm has the inventory option.

The firm value for all three cases as a function of investment size K is shown in Figure 5. We note that the optimal K for the myopic and strategic firm is significantly larger than that of the myopic firm (9.4 versus 7.6 units of capital), as expected from the above discussion; note that there is no difference in optimal size between the myopic and strategic firms. Clearly, the investment size can be significantly impacted by the ability to maintain output inventory.

We also take a look at how the optimal investment size is affected by certain parameter values and whether the inventory option affects this relationship. In particular, we focus on the following parameters: inventory holding cost k, demand volatility σ, and demand elasticity γ. These results are shown in Figure 6. Not surprisingly, the benchmark firm's K^* is unaffected by k, whereas K^* for both myopic and strategic firm are decreasing in k. The strategic firm's K^* converges to that of the benchmark firm as k is increases sufficiently; however, the myopic firm's K^* can be smaller than that of the benchmark firm for a high enough k. As discussed in Section 3.2, the myopic firm value can be below the benchmark firm's value if k is high enough because it maximizes profit instead of value. For the same reason, the myopic firm will choose a smaller size than the benchmark firm for large enough k, as shown in Figure 6(a).

Next, as Figure 6(b) shows, the benchmark K^* is independent of σ; this is because it has no embedded options, hence the decision is unaffected by volatility. However, both myopic and strategic firms have the inventory option, hence the value increases with volatility, thus K^* in both cases is increasing in σ; we also note in Figure 6(b) that the

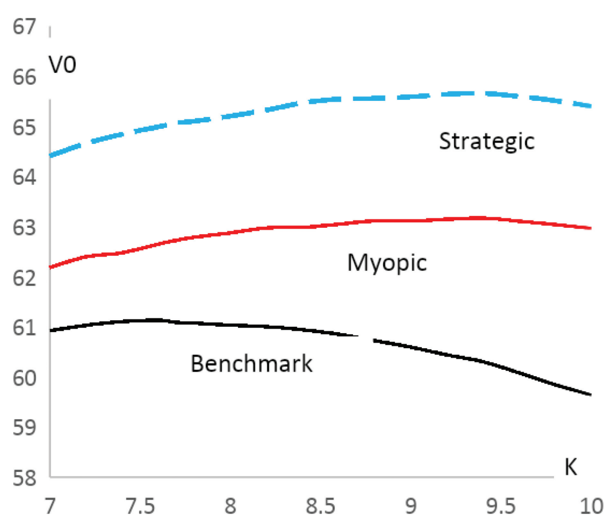

Figure 5: Shows firm value V_0 in all three cases as a function of investment size K by using the base case parameter values: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, $T = 10$, and $\theta_0 = 6$. The optimal investment size is 7.6 for the benchmark firm and 9.4 for both the myopic firm and the strategic firm.

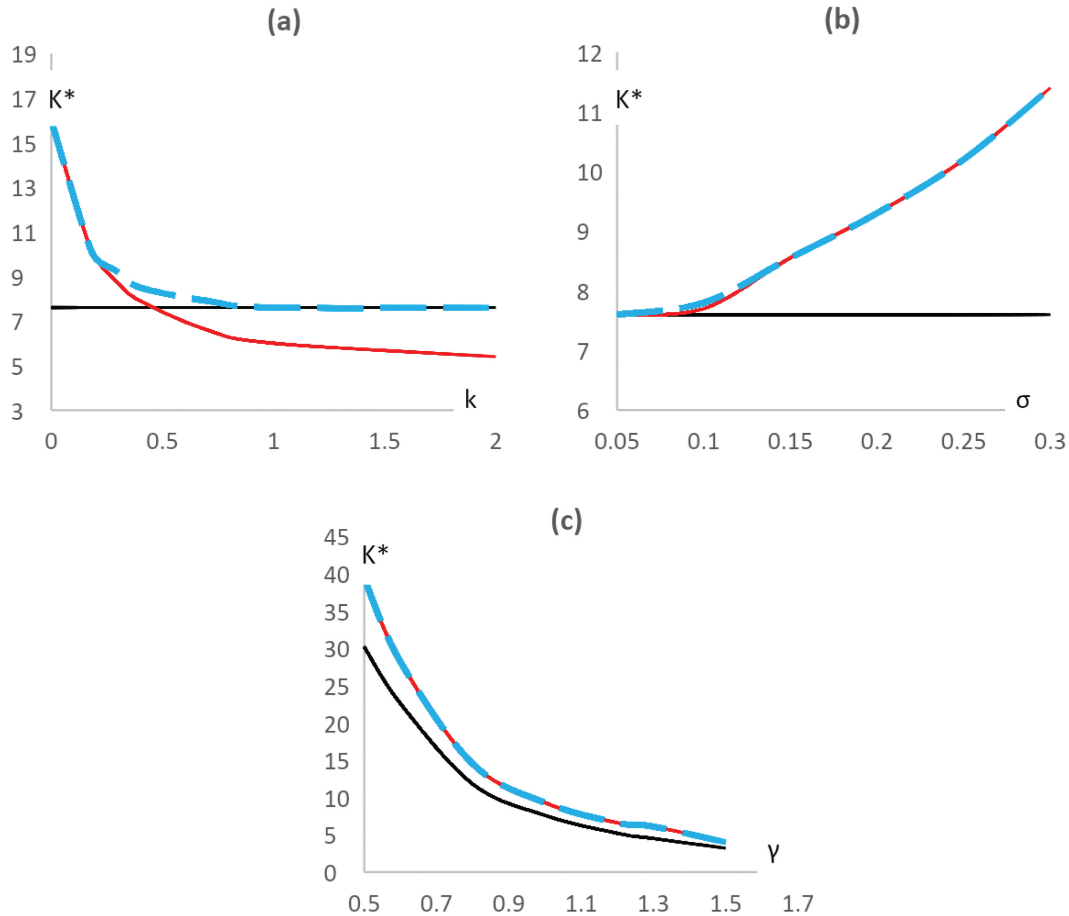

Figure 6: Shows optimal investment size K^* as a function of inventory holding cost k (part a), demand volatility σ (part b) and demand elasticity γ (part c). The base case parameter values are used: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, $\mathrm{T} = 10$, and $\theta_0 = 6$.

optimal size is virtually identical for myopic and strategic firms for all σ. Finally, Figure 6(c) shows the effect of price elasticity γ. A larger γ means that greater output will be relatively less attractive because the price decline will be larger. It is then not surprising that K^* is decreasing in γ for all three firms. Once again, there is no difference in K^* for the myopic and strategic firms. Overall, it seems that there is no difference between optimal investment size for the myopic firm and the strategic firm, although there might be substantial differences in the valuation of the two.

4. Conclusion

This paper studied the contingent-claim valuation of a company that can maintain an inventory of its output. Existing models ignore the possibility of maintaining inventory. We show that the value of a company by following the optimal inventory policy can be significantly higher than the traditional non-inventory company, particularly if the inventory-holding cost is not large. This premium shrinks as holding cost is increased and disappears for large enough holding cost, and is particularly large when demand is volatile, when demand level is low, and when price elasticity is large. Thus, the ability to maintain inventory could potentially have a significant impact on the company's investment and financing decisions.

It is also shown that, when the company's inventory policy is set myopically so as to maximize the current profit rather than long-term value (as is often the case in practice), it is possible that the firm's value actually falls below the traditional no-inventory firm, that is, the premium turns negative. We also show that the optimal investment size for a firm that follows the optimal inventory policy can be significantly larger than the traditional no-inventory firm, particularly when the inventory-holding cost is low, demand volatility is high, and price elasticity is low.

This paper heralds new directions on the application of artificial intelligence (AI) in the dynamic valuation problem. For example, Liu et al. (2023) develops a deep learning-based numerical method (The Seven League scheme) on graphics processing units. Their method improves the computational and convergence speed for large-scale Monte Carlo simulation on stochastic differential equations. Rhijn et al. (2023) have developed generative adversarial networks algorithm to solve stochastic differential equations. They found that the supervised generative adversarial networks outperformed the Euler and Milstein schemes in strong error on a discretization with large time steps. However, their methods can only be applied to well-defined economic boundaries, whereas our model is built on an undefined inventory process, which needs to be optimized along with the iterative valuation. It is obvious that a longer time series simulation with optimized inventory management will lead to more accurate assets valuation. Therefore, the above-mentioned deep leaning algorithm will have great potential on thus topics.

Appendix A: Valuation of Myopic Firm

In the simulations, we need to transform previous continuous time modelling of profit and inventory dynamics to a discrete-time setting. The discrete approximation to Equation (1) is as follows:

$$\theta_{i,j} = \theta_{i,j-1} \exp\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_{i,j} \tag{A.1}$$

where the subscript i represents the i^{th} path and j denotes the time period, the profit flow thereby can be rewritten as

$$\pi_{i,j} = \left(\theta_{i,j} - \gamma q_{i,j}\right)q_{i,j} - cQ - kN_{i,j} \tag{A.2}$$

The current inventory $N_{i,j}$ can be expressed as

$$N_{i,j} = N_{i,j-1} + \left(Q - q_{i,j}\right)\Delta t \tag{A.3}$$

Note that $N_{i,j-1}$ is the inventory stock at previous period.

The instantaneous profit at time spot j can be rewritten as

$$\pi_{i,j} = \left(\theta_{i,j} - \gamma q_{i,j}\right)q_{i,j} - cQ - k\left(N_{i,j-1} + \left(Q - q_{i,j}\right)\Delta t\right) \tag{A.4}$$

with the following non-negative requirement

$$min\left(N_{i,j}, N_{i,j-1}\right) \geq 0 \tag{A.5}$$

The firm value will be calculated by

$$V_{i,j} = \exp\left(-r\Delta t\right) V_{i,j+1} + \int_0^{\Delta t} \pi_{i,j} \tag{A.6}$$

The last item is given by

$$\int_0^{\Delta t} \pi_{i,j} = \left(\left(\theta_{i,j} - \gamma q_{i,j}\right)q_{i,j} - cQ - kN_{i,j-1}\right)\Delta t - \int_0^{\Delta t}\left(k\int_0^x \left(Q - q_{i,j}\right)d\tau\right)dx \tag{A.7}$$

And it can be further written as

$$\int_0^{\Delta t} \pi_{i,j} = \left(\left(\theta_{i,j} - \gamma q_{i,j}\right)q_{i,j} - cQ - kN_{i,j-1}\right)\Delta t - \frac{1}{2}k\left(Q - q_{i,j}\right)\Delta t^2 \tag{A.8}$$

Note the last integrand captures the cumulative inventory storage cost over Δt.

Recall that the profit flow over Δt is

$$\int_0^{\Delta t} \left(pi_j | N_{j-1}\right) = \left(\left(\theta_j - \gamma q_j\right)q_j - cQ - kN_{j-1}\right)\Delta t - \frac{1}{2}k\left(Q - q_j\right)\Delta t^2 \tag{A.9}$$

Take first-order condition with respect to q_j, we have optimal sales for the current time spot j

$$q_j = \frac{2\theta_j + k\Delta t}{4\gamma} \tag{A.10}$$

The upper sales amount should be constrained by upper limit $\overline{q}_j\Delta t < N_{j-1} + Q\Delta t$, or we have

$$q_j = \begin{cases} \frac{2\theta_j + k\Delta t}{4\gamma} \; if \; \theta_j < \overline{\theta}_j \; \frac{N_{j-1}}{\Delta t} + Q \; if \; \theta_j \geq \overline{\theta}_j \end{cases} \tag{A.11}$$

here $\overline{\theta}_j$ is the upper demand threshold

$$\overline{\theta}_j = 2\gamma\left(\frac{N_{j-1}}{\Delta t} + Q\right) - \frac{k\Delta t}{2} \tag{A.12}$$

Appendix B: Valuation of the Strategic Firm

The algorithm for the strategic firm is more complex due to the unknown inventory decisions. Before presenting optimal solutions to the entire time series in a simulated discrete time periods, we start with presenting a very simple case. We assume that the entire life only has four periods: the demand shocks are $\theta = \theta_t$, for $t = 0$, 1, 2, 3, 4. Note that production begins at $t = 1$. The demand shocks in the time series will be produced in a randomness generator in MATLAB, The MathWorks, Inc., Massachusetts, United States.

The firm's value, as a function of sales at four periods q_1, q_2, q_3, q_4 are

$$V_0 = max_{q_1, q_2, q_3, q_4}\left(e^{-r\Delta t}\Pi_1 + e^{-2r\Delta t}\Pi_2 + e^{-3r\Delta t}\Pi_3 + e^{-4r\Delta t}\Pi_4\right) \tag{A.13}$$

here $\Pi_j = \int_0^{\Delta t} \pi_j$ represents cumulative profit flow over period of Δt.

The firm value can be expended easily in the following for a general case, with subscript j as the j^{th} time period

$$V_0 = max_{q_{j=1,2,3,\ldots T}} \sum_{j=1}^{T} \left\{e^{-jr\Delta t}\left[\left(\theta_j - \gamma q_j\right)q_j - cQ - kN_{j-1}\right]\Delta t - k\left(Q - q_j\right)\Delta t^2/2\right\} \tag{A.14}$$

We list for up to four periods for the purpose of *induction and deduction*. In what follows, we first present detailed solution to the reduced case, then we go to implementation details in general case.

$$\begin{aligned} V_0 = {} & e^{-r\Delta t}\left\{[(\theta_1 - \gamma q_1)q_1 - cQ]\Delta t - k(Q - q_1)\Delta t^2/2\right\} \\ & + e^{-2r\Delta t}\left\{[(\theta_2 - \gamma q_2)q_2 - cQ - kN_1]\Delta t - k(Q - q_2)\Delta t^2/2\right\} \\ & + e^{-3r\Delta t}\left\{[(\theta_3 - \gamma q_3)q_3 - cQ - kN_2]\Delta t - k(Q - q_3)\Delta t^2/2\right\} \\ & + e^{-4r\Delta t}\left\{[(\theta_4 - \gamma q_4)q_4 - cQ - kN_3]\Delta t - k(Q - q_4)\Delta t^2/2\right\} \end{aligned} \tag{A.15}$$

Note that inventory at $t = 0$ is zero, and we need the following constraints

$$\begin{aligned} N_1 &= max\,(Q - q_1, 0)\Delta t \\ N_2 &= max\,[N_1 + (Q - q_2)\Delta t, 0] \\ N_3 &= max[N_2 + (Q - q_3)\Delta t, 0] \\ N_4 &= max\,[N_3 + (Q - q_4)\Delta t, 0] \end{aligned}$$

They will ensure non-negative inventory conditions and regulate the maximum sales at each time. This is a super-large (particularly for full times) constrained optimization problem. Even for the four-period case, a traditional optimization such as the Kuhn-Tucker method with regard to all of q_1, q_2, q_3, q_4, N_1, N_2, N_3, N_4, are difficult to implement here because the max-type function makes the entire value non-differentiable at all domains and it depends on inventory status. We, therefore, adopt an iteration method, overall, our optimization for each simulated path can simply be written as

$$V_0 max_{q_1,\ldots,q_T, Q} = \sum_{i=0}^{T} e^{-ri\Delta t}\Pi_i(q_i, N_{i-1}(q_1, ., q_{i-1})), \tag{A.16}$$

subject to $N_{i=Q, i\in[1,T]} > 0$ and

$$N_{i\neq Q, i\in[1,T]} = 0 \tag{A.17}$$

where Q represents the collection of all non-zero inventories. Note that the challenge here is that the set Q is unknown *ex ante* in the constraints and it has to be solved along with the optimization problem, and we call this *iterative optimization*.

In particular, at initial production $t = 1$, the sales must be less or equal than capacity, for example, $q_1 \leq Q$. Define inventory status as binary: zero or nonzero. We could totally have $2^3 = 8$ scenarios. Notice that the last period inventory N_4 has no impact here. In what follows, we only discuss five sample cases for the sake of exhibition. All other cases can be derived in a similar way:

Scenario 1: $N_1 > 0, N_2 > 0, N_3 > 0$

The first-order condition with respect to q_i are as follows:

$$\frac{\partial V_0}{\partial q_1} = e^{-r\Delta t}(\theta_1 - 2\gamma q_1)\Delta t + \left(e^{-r\Delta t}/2 + e^{-r2\Delta t} + e^{-r3\Delta t} + e^{-r4\Delta t}\right)k\Delta t^2 = 0 \tag{A.18}$$

$$\frac{\partial V_0}{\partial q_2} = e^{-2r\Delta t}(\theta_2 - 2\gamma q_2)\Delta t + \left(e^{-2r\Delta t}/2 + e^{-r3\Delta t} + e^{-r4\Delta t}\right)k\Delta t^2 = 0 \tag{A.19}$$

$$\frac{\partial V_0}{\partial q_3} = e^{-3r\Delta t}(\theta_3 - 2\gamma q_3)\Delta t + \left(e^{-3r\Delta t}/2 + e^{-r4\Delta t}\right)k\Delta t^2 = 0 \tag{A.20}$$

$$\frac{\partial V_0}{\partial q_4} = e^{-4r\Delta t}\left((\theta_4 - 2\gamma q_4)\Delta t + k\Delta t^2/2\right) = 0 \tag{A.21}$$

Then we have

$$q_1^* = \frac{\theta_1 + \left(\frac{1}{2} + e^{-r\Delta t} + e^{-2r\Delta t} + e^{-3r\Delta t}\right)k\Delta t}{2\gamma} \tag{A.22}$$

$$q_2^* = \frac{\theta_2 + \left(\frac{1}{2} + e^{-r\Delta t} + e^{-2r\Delta t}\right)k\Delta t}{2\gamma} \tag{A.23}$$

$$q_3^* = \frac{\theta_3 + \left(\frac{1}{2} + e^{-r\Delta t}\right)k\Delta t}{2\gamma} \tag{A.24}$$

$$q_4^* = min\left(\frac{\theta_4 + k\Delta t/2}{2\gamma}, Q + \frac{N_3}{\Delta t}\right) \tag{A.25}$$

Therefore, we have optimal sales at each instant time $q_j^* = \frac{\theta_j + \left(\sum_{i=j}^{T} e^{-r(T-i)\Delta t} - \frac{1}{2}\right)k\Delta t}{2\gamma}$, the idea is simple: each optimal sales equals demand shock plus all current and future savings on the inventory storage. Moreover, it is interesting that each optimal sales is independent of future demand shocks. However, the precondition is that inventories at all periods should be positive, that is,

$$N_j = N_{j-1} + \left(Q - q_j\right)\Delta t > 0 \tag{A.26}$$

Actually this condition also equivalently regulates the upper sales quantities

$$\overline{q}_j \Delta t < N_{j-1} + Q\Delta t \tag{A.27}$$

or an upper-demand threshold beyond which the firm will clear inventories, which is obtained by substituting previous solutions on the optimal sales

$$\overline{\theta}_j < 2\gamma\left(\frac{N_{j-1}}{\Delta t} + Q\right) - \left(\sum_{i=j}^{T} e^{-r(T-i)\Delta t} - \frac{1}{2}\right)k\Delta t \tag{A.28}$$

For simplicity, we will not discuss the negative profits, which will generate lower sales (demand) boundaries $\underline{q}_j\left(\underline{\theta}_j\right)$ below which the sales will be zero. In fact, it is doable, for example, substitute the optimal sales solution back to profit function and solve the positive root because the quadratic equation will be convex shaped. However, the multiple switching thresholds will make our algorithm very messy. In what follows, we discuss when the precondition of all positive inventories is violated:

Scenario 2 : $N_1 = N_2 = N_3 = 0$

$$q_1^* = q_2^* = q_3^* = Q,\ q_4^* \text{ is same as in Scenario 1}$$

Scenario 3 : $N_1 = 0, N_2 > 0, N_3 > 0$

$$q_1^* = Q,\ q_2^*,\ q_3^*,\ q_4^* \text{ are same as in Scenario 1}$$

Scenario 4 : $N_1 > 0,\ N_2 = 0,\ N_3 > 0$

This scenario is a little more complicated. Because, when $N_2 = 0$, it means we could not freely optimize q_2^*, instead, q_2^* is constrained by $q_2^* = 2Q - q_1^*$. So, this case becomes a constrained optimization problem.

$$max\, V_0(q_1, ., q_4),\ \text{subject to}\ 2Q - q_1 - q_2 = 0 \tag{A.29}$$

So we have to re-substitute the updated q_2^* to optimize q_1^*:

$$q_1^* = \frac{\theta_1 - e^{-r\Delta t}\theta_2 + 4\gamma Q e^{-r\Delta t} + k\Delta t(1 + e^{-r\Delta t})/2}{2\gamma(1 + e^{-r\Delta t})} \tag{A.30}$$

This equation can be further rewritten as

$$q_1^* = \underbrace{\frac{\theta_1 + k\Delta t/2}{2\gamma}}_{\text{myopic effect}} + \underbrace{\left(2Q - \frac{\theta_1 + \theta_2}{2\gamma}\right)\frac{e^{-r\Delta t}}{1 + e^{-r\Delta t}}}_{\text{inventory effect}} \tag{A.31}$$

This equation has two parts: the first part is the same as the myopic case. The second part captures the inventory effect: for example, when the future demand θ_2 increases, then q_1^* decreases, that is, the previous sales should decrease to leave some inventory for future sales, which is intuitive because the future price will become higher. The affiliated item $\frac{e^{-r\Delta t}}{1+e^{-r\Delta t}}$ can be considered as the "weight ratio" to capture the discount weight of future effect. Last, q_3^* and q_4^* are the same as in Scenario I.

Scenario 5 : $N_1 > 0,\ N_2 > 0,\ N_3 = 0$

The logic is similar to Scenario 4. We have freedom to optimally select q_1^* and q_2^*, while leaving the equality constraint $q_3^* = 3Q - q_1^* - q_2^*$. We substitute this constrain to the value function and take the first derivative to both q_1 and q_2:

$$q_2^* = \frac{\theta_2 + k\Delta t/2}{2\gamma} + \left(3Q - q_1^* - \frac{\theta_2 + \theta_3}{2\gamma}\right)\frac{e^{-r\Delta t}}{1 + e^{-r\Delta t}} \tag{A.32}$$

and at the first period

$$q_1^* = \frac{\theta_1 + g(k\Delta t/2)}{2\gamma} + \left(3Q - q_2^* - \frac{\theta_1 + \theta_3}{2\gamma}\right)\frac{e^{-2r\Delta t}}{1 + e^{-2r\Delta t}} \tag{A.33}$$

here, the coefficient

$$g = 1 + \frac{2e^{-r\Delta t}}{1 + e^{-2r\Delta t}} \tag{A.34}$$

Obviously, we can solve the two-variable equations for q_1^* and q_2^*, however, when there are many periods until the inventory encounters zero at the first time (e.g., the constraint becomes $q_j^* = iQ - \sum_{z=1}^{i-1} q_{z\neq j}^*$), we cannot do this. For example, suppose $N_1,\ N_2, N_3, \ldots, N_{i-1} > 0$ and $N_i = 0$ and let us derive q_1^*, q_2^*, q_3^*,…, q_{i-1}^*:

$$q_j^* = \frac{\theta_j + g_j(k\Delta t/2)}{2\gamma} + \left(iQ - \sum_{z=1}^{i-1} q_{z\neq j}^* - \frac{\theta_j + \theta_i}{2\gamma}\right)\frac{e^{-(i-j)r\Delta t}}{1 + e^{-(i-j)r\Delta t}} \tag{A.35}$$

The coefficient g_j can be expressed as

$$g_j = \begin{cases} 1 + \dfrac{2\sum\limits_{v=1}^{i-2} e^{-vr\Delta t}}{1 + e^{-(i-j)r\Delta t}} & \text{for } j > i-1 \\ 1 & \text{for } j = i-1 \end{cases} \tag{A.36}$$

The equation system can be rewritten as

$$\boldsymbol{A}_{(\boldsymbol{i}-1)\times(\boldsymbol{i}-1)} q^*_{(\boldsymbol{i}-1)\times 1} = \boldsymbol{B}_{(\boldsymbol{i}-1)\times 1} \tag{A.37}$$

where the bold matrices are as following

$$q^*_{(\boldsymbol{i}-1)\times 1} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_{i-1} \end{bmatrix} \tag{A.38}$$

$$\boldsymbol{B}_{(\boldsymbol{i}-1)\times 1} = \begin{bmatrix} \dfrac{\theta_1 + g_1(k\Delta t/2)}{2\gamma} + \left(iQ - \dfrac{\theta_1 + \theta_i}{2\gamma}\right)\dfrac{e^{-(i-1)r\Delta t}}{1 + e^{-(i-1)r\Delta t}} \\ \dfrac{\theta_2 + g_2(k\Delta t/2)}{2\gamma} + \left(iQ - \dfrac{\theta_2 + \theta_i}{2\gamma}\right)\dfrac{e^{-(i-2)r\Delta t}}{1 + e^{-(i-2)r\Delta t}} \\ \vdots \\ \dfrac{\theta_{i-1} + g_{i-1}(k\Delta t/2)}{2\gamma} + \left(iQ - \dfrac{\theta_{i-1} + \theta_i}{2\gamma}\right)\dfrac{e^{-r\Delta t}}{1 + e^{-r\Delta t}} \end{bmatrix} \tag{A.39}$$

$$
\boldsymbol{A}_{(\boldsymbol{i}-1)\times(\boldsymbol{i}-1)} = \begin{bmatrix} 1 & a_1 & a_1 & \cdots & a_1 \\ a_2 & 1 & a_2 & \cdots & a_2 \\ a_3 & a_3 & \ddots & a_3 & a_3 \\ \vdots & a_4 & a_4 & 1 & \vdots \\ a_{i-1} & a_{i-1} & a_{i-1} & \cdots & 1 \end{bmatrix} \quad \text{(A.40)}
$$

here $a_j = \frac{e^{-(i-j)r\Delta t}}{1+e^{-(i-j)r\Delta t}}$ is the last coefficient in the equation. To solve the linear system, we use the Gauss-Seidel method.

Finally, suppose we simulate M paths, the above calculation will be repeated M times. This simple four-period case informs some intuitions:

1. The optimal sales (or inventory) level at the current time t depends on both the previous and future optimal sales (or inventory) level, which should be difficult to produce a closed form solution.
2. Luckily, all future storing costs of optimal sales expression will cut off at the first timing of zero inventories.
3. The solutions to optimal sales can be converted to solutions to determine the optimal timing of depleting inventories, which may still be impossible to solve (for N time periods, we will have 2^N cases of inventories status (e.g., zero or positive)).

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